Asian Journal of Statistical Sciences
Vol. 2, No-2, (2022), pp. 111-127

ARF INDIA
Academic Open Access Publishing
Journal homepage link:
http://arfjournals.com/ajss

Bootstrap Confidence Intervals of the Stress-strength Index for Exponential Populations

Tulika Rudra Gupta[a] and Somesh Kumar[b]

[a,b] *Department of Mathematics, Indian Institute of Technology Kharagpur, 721302, India*

ARTICLE HISTORY
Compiled March 29, 2023

Received 11 January 2023; Accepted 28 February 2023

ABSTRACT
Measures of stress-strength reliability are most frequently used tools in industrial applications. In multi-component systems, it is of interest to compare stress or strength of individual components with that of the average distribution. For this, recently Gupta et al. [14] have introduced a new stress-strength index (SSI) and obtained some point estimators when underlying populations have exponential distributions. In this paper, we develop bootstrap confidence intervals for SSI and show that they have superior performance as compared to the asymptotic confidence interval based on the maximum likelihood estimator. The results are illustrated by implementation on a real data set. 'R' packages have been developed and shared on open source platform for applications by users.

KEYWORDS
Average length; Bayes estimators; Coverage probability; Maximum likelihood estimator; Percentile bootstrap confidence interval; Uniformly minimum variance unbiased estimator

1. Introduction

Let X and Y be two independent random variables, where X and Y denote the stress and the strength respectively of a system. In reliability and life-testing literature, the term $\xi = P(X < Y)$ is referred to as the stress-strength reliability (SSR). In fact, ξ was first introduced in nonparametric inference where locations of two populations were to be compared. The tests based on ranks e.g., Wilcoxon-Mann-Whitney tests (Wilcoxon [34], Mann and Whitney [23]) essentially use evaluation of ξ. However, SSR has rapidly found applications in reliability estimation in diverse problems in engineering design, survival analysis and socio-economic studies. Guttman et al. [14] have considered reliability of a rocket motor case so that it can withstand operating pressure. An application in measuring shear strength for spot welds for two different gauges of steel is also studied. Surles and Padgett [32] have applied this concept to analyze a carbon fibre strength data set. A specially designed three wheeled motorized vehicle called 'motorette' was introduced in USA after world war II for use in congested localities. Gupta et al. [13] considered estimation of ξ for a data set on

CONTACT Somesh Kumar [b]. Email: smsh@iitkgp.ac.in

20 motorettes which run under certain temperature (see Nelson [26], page 115)

Inferential procedures on SSR for several parametric models e.g. Weibull, normal, pareto, gamma, exponential have been developed by several authors, see see Kelley et al. [19], Woodward and Kelley [35], Hanagal [15], Weerahandi and Johnson [33], Asgharzadeh et al. [4]). One may refer to Kotz et al. [20] for a detailed literature review on the work on stress-strength reliability.

In design of large and complex engineering systems, one needs to extend the definition of SSR for estimating true reliability of the given system. Suppose, a component with strength X is subjected to independent stresses $Y_1, \ldots, Y_k$. Then, SSR for such a component is defined as $\xi_1 = P(X > max(Y_1, \ldots, Y_k))$. Estimation of ξ_1, when stress and strength distributions are exponential, has been studied by Jana et al. [18], Kundu et al. [22]. Some equipment function smoothly as long as the temperature is maintained between two limits. The reliability of such equipment can be defined as $\xi_2 = P(X_1 < Y < X_2)$, where X_1 and X_2 denote lower and upper temperature limits for the equipment to operate, (see see Singh [31] for an application). A further extension of the measure ξ_2 is to consider $\xi_3 = P(X_1 < X_2 < \ldots < X_k)$. This is quite useful in the study of isotonic regression (Miwa et al. [24]).

In the design of large electrical or electronic circuits, a common practice is to use components connected in series or parallel structures. Let lives of k components be denoted by $X_1, \cdots, X_k$ respectively. Then $U = min(X_1, \cdots, X_k)$ denotes the life of the series system, whereas $V = max(X_1, \cdots, X_k)$ will be the life of the parallel system. Mori and Kato [25] and Patelli et al. [27] have developed importance sampling methods for evaluating reliability of series and parallel systems. A measurement of a few explanatory variables that has an impact on the strength or the stress is frequently available to the experimenter. Guttman et al. [14] and Aminzadeh [3] have considered estimation of SSR in the presence of explanatory variables.

There may be a situation where the system may be in more than two (success and failure) states. For example, in progression of a disease, decay of an organism etc. Eryilmaz [8, 9, 10, 11] extend the traditional stress-strength reliability to model such problems. Eryilmaz [12] further extended this notion to model SSR for multi-state k-out-of-n system. Eryilmaz studied properties of his measure for several probability models. Inference problems on the SSR measure introduced by Eryilmaz have been investigated in detail by Jana et al. [16, 17], Qin et al. [28] and Jana et al. [18] when underlying stress and strength random variables follow exponential distributions. The maximum likelihood estimators (MLEs), uniformly minimum variance unbiased estimators (UMVUEs), the best equivariant estimators, Bayes and generalized Bayes estimators have been derived for each model and their performance compared numerically. In some cases, bootstrap based confidence intervals have also been constructed. Some industrial systems contain components in active, warm standby and cold standby states. Siju and Kumar [30] and Kumar and Siju [21] have obtained expressions for reliability of such systems when lives of components follow exponential, Weibull or Lomax distributions. MLEs, Bayes estimators and bootstrap confidence intervals have been derived.

Asymptotic and bootstrap confidence intervals based on MLE for SSR have been obtained and compared in Al-Mutairi et al. [1] when stress and strength follow

Lindley distributions. Al-Mutairi et al. [2] extended this work on confidence intervals for SSR to weighted Lindley distributions. Azhad et al. [5] have considered point and confidence interval estimation of multi-component SSR for Pareto distribution based on upper record values.

A further extension of the basic SSR measure ξ has been recently introduced in Gupta et al. [14]. In a system consisting of several components, the stress (strength) of each component relative to that of the average distribution is an important characteristic. Let $F_1, \ldots, F_k$ denote the cumulative distribution functions of stresses (strengths) of k different components and let $H(x) = \frac{1}{k}\sum_{i=1}^{k} F_i(x)$ be their unweighted mean. Gupta et al. [14] defined a new stress-strength index (SSI) of the ith component as

$$P_i = \int F_i dH, \; i = 1, \ldots, k. \tag{1}$$

A value of P_i less than 0.5 indicates that the stress (strength) levels of the ith group tend to be lower than those from the average distributions. The measure P_i has its origins in a measure of the relative treatment effect considered in nonparametric ANOVA models (see Brunner and Puri [7] and Brunner et al. [6]).

In this paper, we develop confidence intervals for SSIs of k groups when the underlying distributions for all groups are exponential with different scale parameters. In Section 2, we have described SSIs for one parameter exponential distribution and introduce various estimators as obtained in Gupta et al. [14]. In Section 3, a parametric bootstrap approach is used to develop confidence intervals for SSIs using percentile method. The average lengths and coverage probabilities for SSIs are evaluated numerically and presented in Section 3 for various configurations of sample sizes. We have given a detailed study for $k = 3$ and 4 and also developed 'R' packages for applications to real data sets. The 'R' package has been shared on open platform 'GitHub'. In Section 4, we have taken a real data set to show implementation of the package.

2. Model Description and Point Estimators of SSI

Suppose $X_{i1}, \ldots, X_{in_i}$ are mutually independent random samples from k exponential populations with unknown scale parameters θ_i, $i = 1, \ldots, k$. The probability density function associated with the ith population is taken to be

$$f_i(x) = \frac{1}{\theta_i} e^{-\frac{x}{\theta_i}}, \; x > 0, \theta_i > 0, \tag{2}$$

and let $F_i(x)$ be the corresponding cumulative distribution function (CDF) for $i = 1, \ldots, k$. We then consider

$$H(x) = \frac{1}{k}\sum_{i=1}^{k} F_i(x) \tag{3}$$

as the mixture of all k CDFs with equal weights. Clearly H is also a CDF and let Y be a random variable with this CDF. Then, SSI for the ith group as defined in (1) can be expressed as $P_i = P(X_{i1} < Y)$. From Equations (2) and (3), SSI for the ith group is obtained as

$$P_i = \left(1 - \frac{1}{2k}\right) - \frac{1}{k}\sum_{j=1, j\neq i}^{k} \frac{\theta_i}{\theta_i + \theta_j},\ i = 1, \ldots, k. \tag{4}$$

Some elementary properties of SSI are given in the following lemma.

Lemma 2.1. *For SSI defined in Equation (4), we have*

(i) $\frac{1}{2k} \leq P_i \leq 1 - \frac{1}{2k}$.

(ii) $\frac{1}{k}\sum\limits_{i=1}^{k} P_i = 0.5$.

Proof. (i) The left hand inequality follows from the fact that $\frac{\theta_i}{\theta_i+\theta_j} < 1$ for all $i, j = 1, \ldots, k$. The right hand inequality is obvious as θ_i's are all positive.
(ii) Combining the term $\frac{1}{2k}$ with the terms in the summation, we can express

$$\sum_{i=1}^{k} P_i = k - \frac{1}{k}\sum_{i=1}^{k}\sum_{j=1}^{k} a_{ij},$$

where $a_{ij} = \frac{\theta_i}{\theta_i+\theta_j}$ for $i, j = 1, \ldots, k$. Note that $a_{ii} = \frac{1}{2}$ for $i = 1, \ldots, k$ and $a_{ij} + a_{ji} = 1$ for $i \neq j$. Therefore $\sum\limits_{i=1}^{k} P_i = k - \frac{1}{k}\left(\frac{k}{2} + \frac{k(k-1)}{2}\right) = \frac{k}{2}$. □

In order to derive confidence intervals for P_i's, p-bootstrap approach is used. For pivot statistics, we utilize point estimators as developed in Gupta et al. [14]. These are the MLE, UMVUE, a generalized Bayes estimator and an analogue of the best scale equivariant (BSE) estimator.

2.1. The MLE

Define $T_i = n_i\bar{X}_i = \sum\limits_{j=1}^{n_i} X_{ij}$, $i = 1, \ldots, k$. The MLE of SSI is

$$\delta_{i,M} = \left(1 - \frac{1}{2k}\right) - \frac{1}{k}\sum_{j=1, j\neq i}^{k} \frac{n_j T_i}{n_j T_i + n_i T_j},\ \text{for}\ \ i = 1, 2, \ldots k. \tag{5}$$

2.2. The UMVUE

For $i, j = 1, \ldots, k;\ i \neq j$ let $V_{ij} = \frac{T_i}{T_j}$ and

$$\delta_{ij,U} = \begin{cases} 1 - \sum_{l=0}^{n_j-1} (-1)^l \ \frac{\binom{n_j-1}{l}}{\binom{n_i+l-1}{l}} \ V_{ij}^l, \ 0 < V_{ij} \leq 1 \\ 1 - \sum_{l=0}^{n_i-2} (-1)^l \ \frac{\binom{n_i-1}{l+1}}{\binom{n_j+l}{l+1}} \ \frac{1}{V_{ij}^{l+1}}, \ V_{ij} > 1 \end{cases} \tag{6}$$

Then the UMVUE of P_i is

$$\delta_{i,U} = \left(1 - \frac{1}{2k}\right) - \frac{1}{k} \sum_{j=1, j \neq i}^{k} \delta_{ij,U}, \quad i = 1, \ldots, k. \tag{7}$$

2.3. Generalized Bayes Estimator

For the squared error loss function

$$L(P_i, a_i) = (P_i - a_i)^2, \quad \text{for } i = 1, \ldots, k \tag{8}$$

Gupta et al. [14] considered an improper prior for $\underline{\theta} = (\theta_1, ..., \theta_k)^T$ as

$$g^*(\underline{\theta}) = \prod_{i=1}^{k} \frac{1}{\theta_i}, \ \theta_i > 0, \ i = 1, 2, ..., k. \tag{9}$$

Let $h_{ij} = \frac{n_i}{n_i + n_j}$ and

$$\begin{aligned} d_{1,n_i,n_j,V_{ij}} &= {}_2F_1(n_j + 1, n_i + n_j, n_i + n_j + 1, 1 - \frac{1}{V_{ij}}), \\ d_{2,n_i,n_j,V_{ij}} &= {}_2F_1(n_i, n_i + n_j, n_i + n_j + 1, 1 - V_{ij}), \ i \neq j, \ i, j = 1, \ldots, k. \end{aligned}$$

The generalized estimator of $P_i;\ i = 1, \ldots, k$, with respect to the improper prior (9) was derived as

$$\delta_{i,GB} = \begin{cases} (1 - \frac{1}{2k}) - \frac{1}{k} \sum_{j=1, j \neq i}^{k} \ (\frac{1}{V_{ij}})^{n_j} \ h_{ji} d_{1,n_i,n_j,V_{ij}}, \ \text{if } 0 < V_{ij} \leq 1 \\ (1 - \frac{1}{2k}) - \frac{1}{k} \sum_{j=1, j \neq i}^{k} \ {V_{ij}}^{n_i} \ h_{ji} \ d_{2,n_i,n_j,V_{ij}}, \ \text{if } V_{ij} > 1. \end{cases} \tag{10}$$

2.4. An Analogue of BSE Estimator

Gupta et al. [14] also proposed an analogue of best scale equivariant estimator of P_i as

$$\delta_{i,BS} = \left(1 - \frac{1}{2k}\right) - \frac{1}{k} \sum_{j=1, j \neq i}^{k} \frac{n_j + 1}{(n_j + 1) + (n_i - 2) V_{ji}}, \ i = 1, \ldots, k. \tag{11}$$

2.5. Asymptotic Distribution of MLE

Let $\underline{P} = (P_1, \ldots, P_k)^T$ and $\underline{\delta}_M = (\delta_{1,M}, \ldots, \delta_{k,M})^T$. Further, let $N = \sum_{i=1}^{k} n_i$ and assume $\frac{n_i}{N} \to \epsilon_i$ as $min\ n_i \to \infty$, for $i = 1, \ldots, k$. Then the asymptotic distribution of $\sqrt{N}(\underline{\delta}_M - \underline{P})$ is multivariate normal with the mean vector $\underline{0}$ and dispersion matrix B, where B is given by $B = (b_{ij})_{k\times k}$. Here,

$$b_{ii} = \frac{1}{k^2}\left[\sum_{l=1,l\neq i}^{k} \frac{1}{\epsilon_l}\frac{\theta_l^4}{(\theta_i+\theta_l)^4} + \left(\sum_{l=1,l\neq i}^{k} \frac{\theta_l}{(\theta_l+\theta_i)^2}\right)^2 \frac{\theta_i^2}{\epsilon_i}\right] \tag{12}$$

and

$$\begin{aligned} b_{ij} = \frac{1}{k^2}\Bigg[&\sum_{l=1,l\neq i\neq j}^{k} \frac{\theta_l^2}{(\theta_l+\theta_i)^2(\theta_l+\theta_j)^2}\frac{\theta_l^2}{\epsilon_l} - \sum_{l=1,l\neq i}^{k} \frac{\theta_l}{(\theta_l+\theta_i)^2}\frac{\theta_i}{(\theta_i+\theta_j)^2}\frac{\theta_i^2}{\epsilon_i} \\ &- \sum_{l=1,l\neq j}^{k} \frac{\theta_l}{(\theta_l+\theta_j)^2}\frac{\theta_j}{(\theta_j+\theta_i)^2}\frac{\theta_j^2}{\epsilon_j}\Bigg],\ i\neq j,\ i,j = 1,\ldots,k. \end{aligned} \tag{13}$$

In the next section, we will compare the estimated coverage probability (CP) and average length (AL) of confidence interval (CI) based on the above asymptotic distribution of MLE with those based on p-bootstrap confidence intervals.

3. Confidence Intervals

In this section, we develop parametric bootstrap confidence intervals based on estimators described in Section 2. Percentile method is used to construct these intervals based on estimators as pivot statistics. The algorithm for finding parametric bootstrap confidence intervals for P_i based on an estimator δ_i, $i = 1, \ldots, k$ is as follows.

Algorithm:
Step 1: Generate observations X_{ij} from $Exp(\frac{1}{\theta_i})$, for $j = 1, \ldots, n_i$, and $i = 1, \ldots, k$.
Step 2: Calculate corresponding estimators δ_i for ith populations.
Step 3: Generate bootstrap samples from the estimated populations $Exp(\frac{1}{\hat{\theta}_i})$, where $\hat{\theta}_i = \bar{X}_i$, $i = 1, \ldots, k$.
Step 4: Compute the value of the estimator δ_i from the above bootstrap sample, call it δ_i^*.
Step 5: Repeat Steps 3 and 4 'H' number of times to get $\delta_{i1}^*, \ldots, \delta_{iH}^*$ and sort all values of the bootstrap estimators as $\delta_{i(1)}^* \leq \cdots \leq \delta_{i(H)}^*$.
Step 6: The $(1-\alpha)$ p-bootstrap confidence interval for P_i is obtained as

$$\left(\delta^*_{i([\frac{\alpha}{2}H])},\ \delta^*_{i([(1-\frac{\alpha}{2})H])}\right),\ i = 1,\ldots,k.$$

We have computed estimated coverage probabilities and average lengths of p-bootstrap confidence intervals based on estimators $\delta_{i,M}, \delta_{i,U}, \delta_{i,GB}$ and $\delta_{i,BS}$ for various choices of sample sizes and parameters. The nominal confidence level has been chosen to be 0.95 in all tables. For the purpose of comparison, we have also computed the estimated coverage probabilities and average lengths of confidence intervals based on the asymptotic distribution of MLE. Note that P_i's are functions of $\eta_i = \frac{\theta_{i+1}}{\theta_1}$, $i = 1, 2, ..., k-1$. For $k = 3$, when $\eta_1 = 1$, we have taken (n_1, n_2, n_3)=$(5, 5, 10)$, and when $\eta_1 = 0.5$, choice of sample sizes are (n_1, n_2, n_3)=$(15, 20, 10)$. Similarly, for $k = 4$ and for fixed $\eta_1 = 0.5, \eta_2 = 0.5$, we have considered $(n_1, n_2, n_3, n_4) = (5, 5, 6, 10)$, whereas when fixed $\eta_1 = 1, \eta_3 = 2$, the sample size we have taken as $(n_1, n_2, n_3, n_4) = (30, 25, 20, 15)$. In this study, we have used 1000 replications of simulation step and 300 iterations of bootstrap sampling for each replication.

In Tables 1-14, coverage probabilities and average lengths of confidence intervals based on estimators $\delta_{i,M}, \delta_{i,U}, \delta_{i,GB}$, $\delta_{i,BS}$ and the asymptotic distribution of MLE are captioned as 'MLE', 'UMVUE', 'GB', 'BSE' and 'AMLE' respectively. Observations on the comparative performance of CPs and ALs of various CIs based on Tables 1-14 are presented in the following remarks.

Table 1. CP and AL of CI for P_1: $k = 3$, $n_1 = 15, n_2 = 20, n_3 = 10$ and $\eta_1 = 0.5$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.957	0.939	0.941	0.816	0.917	0.1481	0.1502	0.1446	0.1356	0.1268
0.4	0.939	0.941	0.959	0.823	0.919	0.1729	0.1769	0.1699	0.1612	0.1517
0.6	0.955	0.947	0.951	0.838	0.9	0.1842	0.1891	0.1808	0.1748	0.1685
0.8	0.932	0.949	0.94	0.791	0.918	0.1895	0.1957	0.1851	0.1799	0.1802
1	0.958	0.936	0.961	0.808	0.932	0.1913	0.1959	0.1862	0.1845	0.1958
1.2	0.941	0.948	0.944	0.811	0.938	0.1911	0.1968	0.1859	0.1859	0.2045
1.4	0.95	0.958	0.953	0.811	0.95	0.1902	0.1956	0.1843	0.1862	0.2145
1.6	0.95	0.949	0.958	0.819	0.966	0.1887	0.1921	0.1826	0.1863	0.2262
1.8	0.952	0.944	0.947	0.805	0.965	0.1865	0.1904	0.1808	0.1853	0.2312
2	0.947	0.944	0.951	0.824	0.966	0.1842	0.1874	0.1787	0.1845	0.2387
2.2	0.955	0.944	0.95	0.823	0.967	0.1813	0.1851	0.1766	0.1824	0.2468
2.4	0.94	0.942	0.947	0.821	0.978	0.1791	0.1817	0.1741	0.1807	0.2543
2.6	0.956	0.955	0.944	0.818	0.973	0.1771	0.1804	0.1716	0.1798	0.2585
2.8	0.943	0.944	0.949	0.794	0.982	0.1744	0.1769	0.1702	0.1765	0.2646
3	0.951	0.944	0.933	0.811	0.986	0.1726	0.1742	0.1702	0.1751	0.2709
3.2	0.942	0.939	0.936	0.785	0.983	0.1691	0.1709	0.1683	0.1729	0.2769
3.4	0.948	0.936	0.921	0.788	0.988	0.167	0.1696	0.1667	0.1715	0.2812
3.6	0.948	0.94	0.94	0.783	0.987	0.1655	0.1678	0.1637	0.1699	0.2857
3.8	0.949	0.953	0.937	0.789	0.988	0.1638	0.1648	0.1614	0.1681	0.2918
4	0.952	0.959	0.942	0.777	0.994	0.1611	0.1636	0.1586	0.1658	0.2949

Remark 1. In Tables 1-3, the CPs and ALs of CIs are presented for P_1, P_2 and P_3 for $k = 3$ when sample sizes are moderate. The values are dependent on η_1 and η_2.

Table 2. CP and AL of CI for P_2: $k = 3$, $n_1 = 15, n_2 = 20, n_3 = 10$ and $\eta_1 = 0.5$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.927	0.95	0.938	0.822	0.959	0.1631	0.1672	0.1606	0.1604	0.1977
0.4	0.941	0.951	0.953	0.824	0.963	0.1783	0.1839	0.1741	0.1794	0.2208
0.6	0.938	0.942	0.929	0.813	0.96	0.1801	0.1847	0.1752	0.1834	0.2387
0.8	0.938	0.943	0.953	0.815	0.977	0.1774	0.1813	0.1721	0.1827	0.2568
1	0.944	0.929	0.948	0.817	0.982	0.1723	0.1754	0.1682	0.1801	0.2709
1.2	0.945	0.934	0.948	0.819	0.982	0.1677	0.1709	0.1644	0.1768	0.284
1.4	0.945	0.941	0.938	0.789	0.989	0.1636	0.1654	0.1591	0.173	0.2979
1.6	0.943	0.951	0.948	0.796	0.992	0.1599	0.1594	0.1559	0.1687	0.3048
1.8	0.937	0.949	0.936	0.778	0.989	0.1565	0.1573	0.1522	0.1669	0.3148
2	0.935	0.95	0.933	0.784	0.995	0.1527	0.1537	0.149	0.1623	0.3213
2.2	0.939	0.955	0.952	0.792	0.993	0.1483	0.1505	0.1459	0.1588	0.3267
2.4	0.947	0.948	0.932	0.776	0.998	0.1474	0.1469	0.144	0.1565	0.3357
2.6	0.938	0.952	0.946	0.743	0.999	0.1437	0.1448	0.1407	0.1554	0.339
2.8	0.938	0.938	0.936	0.789	0.997	0.1418	0.1417	0.1391	0.151	0.3448
3	0.939	0.956	0.929	0.777	0.999	0.1391	0.1392	0.1397	0.1493	0.3514
3.2	0.942	0.936	0.934	0.802	1	0.1364	0.1369	0.1377	0.1467	0.354
3.4	0.934	0.948	0.926	0.791	0.999	0.1352	0.1361	0.1375	0.1449	0.358
3.6	0.946	0.957	0.927	0.778	1	0.1331	0.133	0.1343	0.1431	0.3619
3.8	0.937	0.939	0.926	0.788	0.997	0.132	0.1319	0.1327	0.1425	0.3652
4	0.946	0.956	0.926	0.782	1	0.1299	0.1302	0.1312	0.1396	0.3669

Table 3. CP and AL of CI for P_3: $k = 3$, $n_1 = 15, n_2 = 20, n_3 = 10$ and $\eta_1 = 0.5$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.936	0.943	0.938	0.728	0.999	0.1576	0.1602	0.1579	0.178	0.3141
0.4	0.946	0.939	0.945	0.776	0.973	0.2009	0.2075	0.1969	0.2121	0.2704
0.6	0.936	0.946	0.946	0.807	0.963	0.2145	0.2205	0.2076	0.2159	0.2473
0.8	0.947	0.941	0.948	0.823	0.955	0.2154	0.2223	0.2091	0.2119	0.2302
1	0.951	0.943	0.941	0.82	0.934	0.2125	0.2177	0.2063	0.2038	0.2135
1.2	0.939	0.949	0.94	0.833	0.931	0.2068	0.2131	0.201	0.1958	0.2022
1.4	0.944	0.948	0.941	0.801	0.917	0.2009	0.2046	0.1943	0.1858	0.1897
1.6	0.943	0.95	0.951	0.804	0.913	0.1948	0.1962	0.1886	0.1776	0.1794
1.8	0.94	0.948	0.941	0.807	0.914	0.1889	0.1911	0.182	0.1706	0.1719
2	0.939	0.941	0.948	0.792	0.914	0.1819	0.1911	0.1773	0.1618	0.1645
2.2	0.956	0.943	0.931	0.781	0.901	0.1746	0.1767	0.1712	0.1545	0.1573
2.4	0.937	0.95	0.942	0.754	0.904	0.1682	0.1693	0.1653	0.1482	0.1487
2.6	0.938	0.956	0.937	0.796	0.931	0.1646	0.1652	0.1594	0.1445	0.1447
2.8	0.932	0.951	0.954	0.754	0.897	0.1594	0.1584	0.1504	0.1377	0.1385
3	0.943	0.957	0.932	0.787	0.895	0.1536	0.1527	0.1489	0.1336	0.1317
3.2	0.942	0.944	0.943	0.758	0.904	0.1479	0.1472	0.1436	0.1278	0.1271
3.4	0.955	0.931	0.94	0.776	0.901	0.1429	0.1424	0.1413	0.1245	0.1227
3.6	0.938	0.944	0.937	0.744	0.911	0.1404	0.1392	0.1661	0.1202	0.1183
3.8	0.934	0.947	0.938	0.75	0.889	0.1367	0.1345	0.1314	0.1165	0.1133
4	0.942	0.952	0.939	0.733	0.925	0.131	0.1308	0.126	0.1118	0.1106

So we have fixed η_1 and varied η_2. It is observed that the estimated CPs for MLE, UMVUE and GB are quite close to the nominal confidence level in all cases. However, those for BSE are substantially smaller than the nominal level making them unreliable. Also CPs for AMLE are seen to be larger than the nominal level when η_2 is large and smaller when η_2 is small. It is also noted that ALs for BSE and AMLE are larger than those for MLE, UMVUE and GB. Thus p-bootstrap CIs based on MLE, UMVUE and GB are preferable.

Remark 2. For $k = 3$ and small sample sizes, the CPs and ALs of proposed CIs are tabulated in Tables 4-6 for P_1, P_2 and P_3. It is observed that even for small sample sizes the CPs of p-bootstrap CIs based on MLE, UMVUE and GB achieve nominal confidence level in all cases, while those based on BSE perform very poorly. The CPs of CIs based on AMLE also have quite inconsistent performance. Among MLE, UMVUE

Table 4. CP and AL of CI for P_1: $k = 3$, $n_1 = 5, n_2 = 5, n_3 = 10$ and $\eta_1 = 1$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.941	0.944	0.944	0.55	0.921	0.2636	0.2751	0.2497	0.2241	0.2789
0.4	0.949	0.944	0.938	0.619	0.933	0.2923	0.306	0.2746	0.2561	0.3081
0.6	0.938	0.934	0.93	0.613	0.936	0.3006	0.3172	0.2817	0.2731	0.3246
0.8	0.933	0.953	0.944	0.574	0.945	0.3015	0.323	0.2855	0.2805	0.3388
1	0.936	0.942	0.947	0.593	0.93	0.3012	0.3221	0.2845	0.2861	0.3425
1.2	0.952	0.948	0.947	0.616	0.946	0.3001	0.3198	0.2837	0.2914	0.3521
1.4	0.945	0.946	0.956	0.564	0.957	0.2956	0.3167	0.2827	0.2891	0.3591
1.6	0.947	0.942	0.942	0.589	0.951	0.2955	0.3138	0.2809	0.2918	0.3628
1.8	0.943	0.948	0.954	0.561	0.962	0.291	0.3101	0.2774	0.2892	0.3658
2	0.944	0.944	0.943	0.565	0.967	0.2862	0.3063	0.2739	0.2913	0.3688
2.2	0.945	0.944	0.942	0.561	0.958	0.2817	0.303	0.2715	0.2893	0.3702
2.4	0.95	0.934	0.937	0.568	0.964	0.2796	0.2994	0.2693	0.2892	0.3757
2.6	0.95	0.952	0.94	0.558	0.967	0.2774	0.2952	0.2641	0.2873	0.3797
2.8	0.941	0.943	0.945	0.537	0.975	0.2755	0.2929	0.263	0.2843	0.3832
3	0.949	0.958	0.935	0.562	0.977	0.2741	0.2901	0.2627	0.2823	0.3835
3.2	0.943	0.947	0.945	0.503	0.979	0.2677	0.287	0.2601	0.2803	0.3852
3.4	0.95	0.957	0.943	0.523	0.974	0.2674	0.284	0.2599	0.2812	0.3863
3.6	0.957	0.951	0.935	0.52	0.981	0.2659	0.2809	0.2579	0.2774	0.3893
3.8	0.944	0.948	0.945	0.537	0.975	0.2623	0.2766	0.2571	0.2764	0.3928
4	0.946	0.932	0.948	0.505	0.983	0.2604	0.2744	0.253	0.2757	0.3908

Table 5. CP and AL of CI for P_2: $k = 3$, $n_1 = 5, n_2 = 5, n_3 = 10$ and $\eta_1 = 1$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.946	0.956	0.945	0.576	0.92	0.264	0.2765	0.251	0.2273	0.2797
0.4	0.952	0.946	0.952	0.598	0.944	0.2923	0.3092	0.2738	0.2552	0.3088
0.6	0.945	0.95	0.937	0.625	0.947	0.2998	0.3188	0.2809	0.2737	0.3294
0.8	0.953	0.949	0.952	0.609	0.944	0.3012	0.322	0.2854	0.2812	3356
1	0.933	0.952	0.942	0.594	0.956	0.3004	0.3214	0.2847	0.2862	0.3484
1.2	0.948	0.944	0.942	0.588	0.944	0.3001	0.3199	0.2841	0.2898	0.3505
1.4	0.936	0.957	0.949	0.613	0.955	0.2949	0.3173	0.2817	0.2916	0.3559
1.6	0.941	0.939	0.955	0.571	0.953	0.2947	0.3137	0.2805	0.2912	0.3618
1.8	0.949	0.936	0.957	0.594	0.956	0.2902	0.3089	0.2773	0.2912	0.3626
2	0.945	0.941	0.95	0.561	0.971	0.2872	0.3062	0.2742	0.2905	0.3678
2.2	0.929	0.943	0.933	0.574	0.957	0.2829	0.3014	0.2721	0.29	0.3719
2.4	0.954	0.944	0.938	0.55	0.964	0.2796	0.298	0.2693	0.2881	0.3748
2.6	0.952	0.952	0.943	0.54	0.965	0.2784	0.2974	0.2651	0.2864	0.3764
2.8	0.944	0.941	0.935	0.561	0.965	0.2742	0.2903	0.2634	0.2867	0.3776
3	0.944	0.94	0.941	0.56	0.969	0.2736	0.2896	0.2629	0.2825	0.3824
3.2	0.934	0.955	0.944	0.588	0.968	0.2685	0.285	0.2634	0.2833	0.3841
3.4	0.942	0.945	0.938	0.514	0.97	0.2664	0.2832	0.2582	0.2821	0.3875
3.6	0.956	0.95	0.952	0.53	0.982	0.2644	0.2794	0.2576	0.2786	0.3887
3.8	0.937	0.942	0.958	0.559	0.979	0.2614	0.2767	0.2561	0.2765	0.3892
4	0.946	0.957	0.939	0.523	0.978	0.2606	0.2751	0.2541	0.2759	0.3942

and GB, the ALs of GB are the shortest in most cases with ALs for CIs based on MLE and UMVUE following these. Thus CIs based on GB are most preferred in this case.

Remark 3. In Tables 7-10, the CPs and ALs of CIs for P_1, P_2, P_3 and P_4 are presented for $k = 4$ when sample sizes are taken to be moderate. Here the values of η_1 and η_3 are fixed and η_2 is varied. It can be observed from the tabulated values that estimated CPs of CIs based on MLE, UMVUE and GB are quite close to the nominal confidence level in all cases while those for BSE are substantially smaller. For AMLE the performance is quite inconsistent with estimated CPs being sometimes quite higher and sometimes quite smaller than the nominal level. It is clear that ALs of CIs based on MLE, UMVUE and GB are quite close to each other in almost all cases making them equally preferable choices.

Table 6. CP and AL of CI for P_3: $k = 3$, $n_1 = 5, n_2 = 5, n_3 = 10$ and $\eta_1 = 1$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.925	0.955	0.918	0.498	0.998	0.1874	0.1844	0.1793	0.2204	0.5669
0.4	0.934	0.943	0.924	0.583	0.976	0.2397	0.2473	0.2252	0.2572	0.4501
0.6	0.928	0.952	0.929	0.609	0.951	0.2576	0.2701	0.2428	0.2622	0.3748
0.8	0.95	0.941	0.923	0.668	0.924	0.2629	0.278	0.2486	0.2591	0.3328
1	0.941	0.936	0.938	0.689	0.913	0.2617	0.2784	0.2494	0.2524	0.297
1.2	0.945	0.947	0.939	0.675	0.901	0.259	0.2756	0.2465	0.2416	0.2738
1.4	0.931	0.949	0.938	0.692	0.873	0.2532	0.2693	0.2441	0.2316	0.2476
1.6	0.939	0.959	0.947	0.683	0.887	0.2489	0.2635	0.2401	0.2218	0.2316
1.8	0.941	0.94	0.931	0.673	0.896	0.2403	0.2556	0.2324	0.212	0.2233
2	0.945	0.946	0.959	0.682	0.884	0.2344	0.2499	0.2275	0.2054	0.2081
2.2	0.946	0.944	0.953	0.678	0.88	0.2276	0.2409	0.2245	0.1959	0.1993
2.4	0.945	0.947	0.94	0.672	0.861	0.2203	0.2332	0.2186	0.1882	0.1872
2.6	0.952	0.949	0.953	0.69	0.853	0.2171	0.229	0.2111	0.1846	0.1782
2.8	0.953	0.945	0.954	0.685	0.841	0.2121	0.2194	0.2078	0.1778	0.1709
3	0.952	0.945	0.947	0.648	0.828	0.2074	0.2161	0.2017	0.1675	0.1635
3.2	0.941	0.947	0.952	0.659	0.842	0.2001	0.207	0.2078	0.1645	0.1591
3.4	0.949	0.938	0.96	0.698	0.842	0.1948	0.2006	0.1917	0.1632	0.1535
3.6	0.952	0.941	0.948	0.651	0.847	0.1907	0.1974	0.19	0.1538	0.1475
3.8	0.95	0.941	0.948	0.642	0.827	0.1834	0.1902	0.1861	0.1487	0.1422
4	0.945	0.95	0.933	0.675	0.84	0.1819	0.1881	0.1821	0.1472	0.1352

Table 7. CP and AL of CI for P_1: $k = 4$, $n_1 = 30, n_2 = 25, n_3 = 20, n_4 = 15$ and $\eta_1 = 1, \eta_3 = 2$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.949	0.943	0.95	0.869	0.976	0.1346	0.1361	0.1324	0.1341	0.1803
0.4	0.953	0.933	0.943	0.881	0.97	0.1459	0.1481	0.1449	0.1459	0.1928
0.6	0.952	0.951	0.936	0.866	0.976	0.152	0.1557	0.1499	0.1497	0.1992
0.8	0.946	0.955	0.931	0.874	0.961	0.1544	0.159	0.1532	0.1547	0.2026
1	0.942	0.959	0.951	0.888	0.982	0.1552	0.1586	0.1533	0.1568	0.2096
1.2	0.953	0.94	0.947	0.863	0.985	0.1548	0.1588	0.1531	0.1569	0.2153
1.4	0.952	0.948	0.938	0.862	0.979	0.154	0.1586	0.1519	0.1567	0.2214
1.6	0.945	0.943	0.941	0.846	0.979	0.1528	0.157	0.1512	0.1551	0.2256
1.8	0.949	0.941	0.93	0.873	0.977	0.1509	0.1542	0.1502	0.1546	0.2288
2	0.95	0.958	0.932	0.871	0.987	0.1506	0.1527	0.1482	0.1534	0.2332
2.2	0.955	0.936	0.928	0.852	0.983	0.1486	0.1529	0.1477	0.1525	0.2387
2.4	0.947	0.927	0.932	0.863	0.988	0.1475	0.1502	0.1476	0.1507	0.24
2.6	0.951	0.942	0.94	0.869	0.987	0.1472	0.1502	0.1447	0.1497	0.2451
2.8	0.94	0.927	0.946	0.865	0.995	0.1449	0.1483	0.1446	0.1483	0.2499
3	0.94	0.944	0.943	0.863	0.992	0.1439	0.1472	0.1436	0.1474	0.2514
3.2	0.94	0.956	0.943	0.873	0.99	0.1423	0.1457	0.1416	0.1466	0.2546
3.4	0.946	0.942	0.935	0.847	0.991	0.1421	0.1434	0.1396	0.1453	0.2589
3.6	0.935	0.944	0.937	0.878	0.992	0.1407	0.1441	0.14	0.1445	0.2612
3.8	0.939	0.938	0.932	0.872	0.993	0.1401	0.1432	0.1393	0.1428	0.2622
4	0.937	0.937	0.945	0.85	0.996	0.1388	0.1427	0.1377	0.1422	0.2648

Remark 4. Tables 11-14 show the CPs and ALs of CIs for P_1, P_2, P_3 and P_4 when $k = 4$ and sample sizes are taken to be small. Here the values of η_1 and η_2 are fixed and those of η_3 are varied. Even for small sample sizes it is noted that the estimated CPs of CIs based on the MLE, UMVUE and GB are quite close to the nominal level. However, CPs of BSE are quite smaller and those of AMLE are somewhat inconsistent. ALs of GB are smaller than those of MLE followed by those of UMVUE. This makes CIs based on GB to be the most preferred choice.

Remark 5. Computations of CPs and ALs for various CIs have also been done for values $k = 5$ and $k = 6$, other combinations of sample sizes and other configurations of η_i for $i = 1, \ldots, k-1$. The findings are similar to those mentioned in Remarks 1-4.

Table 8. CP and AL of CI for P_2: $k = 4$, $n_1 = 30, n_2 = 25, n_3 = 20, n_4 = 15$ and $\eta_1 = 1, \eta_3 = 2$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.948	0.94	0.938	0.879	0.976	0.1418	0.1438	0.1395	0.1412	0.1909
0.4	0.954	0.947	0.948	0.874	0.973	0.1543	0.1565	0.1535	0.1535	0.1959
0.6	0.949	0.936	0.932	0.868	0.973	0.1601	0.1634	0.1582	0.1579	0.2039
0.8	0.937	0.936	0.944	0.873	0.969	0.1625	0.1666	0.1609	0.1637	0.208
1	0.938	0.931	0.936	0.861	0.979	0.1629	0.1671	0.1615	0.1656	0.2141
1.2	0.946	0.936	0.938	0.885	0.966	0.1626	0.1667	0.1607	0.1658	0.2178
1.4	0.939	0.929	0.941	0.876	0.978	0.1621	0.1663	0.1611	0.1644	0.2229
1.6	0.942	0.922	0.943	0.882	0.974	0.1606	0.1648	0.1594	0.1638	0.227
1.8	0.949	0.936	0.932	0.876	0.987	0.1588	0.1636	0.1579	0.1629	0.2319
2	0.938	0.928	0.947	0.855	0.973	0.1577	0.1623	0.1566	0.1614	0.2371
2.2	0.951	0.932	0.938	0.869	0.987	0.1568	0.1608	0.1548	0.1606	0.408
2.4	0.959	0.953	0.947	0.868	0.99	0.1546	0.1592	0.1538	0.1594	0.2445
2.6	0.954	0.943	0.939	0.878	0.985	0.1539	0.1575	0.1525	0.1576	0.2483
2.8	0.944	0.936	0.936	0.871	0.987	0.1528	0.1555	0.1519	0.1566	0.2508
3	0.941	0.952	0.931	0.89	0.989	0.1511	0.1555	0.1501	0.1554	0.2532
3.2	0.956	0.945	0.929	0.859	0.985	0.1505	0.1537	0.1485	0.1544	0.2568
3.4	0.94	0.938	0.95	0.832	0.996	0.149	0.1522	0.1477	0.1538	0.2596
3.6	0.936	0.935	0.941	0.871	0.994	0.1478	0.1514	0.1471	0.1525	0.264
3.8	0.951	0.928	0.917	0.851	0.997	0.1473	0.1497	0.1449	0.1509	0.2643
4	0.944	0.929	0.937	0.879	0.993	0.1456	0.1494	0.1459	0.1499	0.2674

Table 9. CP and AL of CI for P_3: $k = 4$, $n_1 = 30, n_2 = 25, n_3 = 20, n_4 = 15$ and $\eta_1 = 1, \eta_3 = 2$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.93	0.943	0.949	0.821	1	0.0941	0.0938	0.0946	0.1022	0.3768
0.4	0.952	0.937	0.931	0.861	0.998	0.1368	0.1438	0.1377	0.1454	0.3132
0.6	0.95	0.941	0.945	0.844	0.992	0.1585	0.1626	0.1578	0.1578	0.2713
0.8	0.958	0.938	0.946	0.857	0.986	0.1699	0.1734	0.1681	0.1744	0.2421
1	0.948	0.921	0.936	0.876	0.951	0.1745	0.179	0.1719	0.1778	0.2179
1.2	0.948	0.923	0.946	0.862	0.958	0.1772	0.1805	0.1735	0.1771	0.2017
1.4	0.95	0.947	0.943	0.882	0.956	0.1769	0.1798	0.1745	0.1756	0.1886
1.6	0.964	0.939	0.931	0.897	0.94	0.1755	0.1783	0.1711	0.1726	0.1757
1.8	0.959	0.928	0.947	0.877	0.94	0.1724	0.1751	0.1691	0.1687	0.1645
2	0.942	0.934	0.943	0.85	0.91	0.1698	0.1721	0.1657	0.1622	0.1559
2.2	0.947	0.939	0.956	0.887	0.915	0.1669	0.1695	0.1639	0.1617	0.1469
2.4	0.945	0.942	0.945	0.869	0.908	0.1633	0.1658	0.1598	0.1563	0.1406
2.6	0.94	0.935	0.935	0.867	0.888	0.1609	0.1611	0.1561	0.1528	0.1339
2.8	0.925	0.946	0.929	0.874	0.898	0.1571	0.1583	0.1537	0.1495	0.1291
3	0.943	0.933	0.949	0.868	0.894	0.1536	0.1546	0.1499	0.1448	0.1231
3.2	0.948	0.94	0.949	0.881	0.884	0.1504	0.1495	0.1467	0.1418	0.1189
3.4	0.945	0.953	0.944	0.886	0.873	0.1468	0.1468	0.1426	0.1383	1153
3.6	0.945	0.94	0.941	0.888	0.865	0.1434	0.1428	0.1404	0.1351	0.1107
3.8	0.946	0.95	0.937	0.872	0.865	0.1411	0.1405	0.1382	0.1314	0.1071
4	0.937	0.936	0.949	0.887	0.859	0.1381	0.137	0.1343	0.1282	0.1028

However, as the sample sizes increase, the ALs of CIs based on MLE, UMVUE and GB are seen to be quite close making them equally preferred choices. Also for the same configurations of parameters, if the sample sizes are increased then the ALs of CIs are seen to decrease. For the sake of space, we have omitted those tables.

Remark 6. We have developed packages in 'R' for evaluating estimators and confidence bounds for real data sets for $k = 3$ and $k = 4$. The packages are uploaded on 'GitHub' by the first author (account name 'TulikaStat') and named 'CIEXP3' and 'CIEXP4'. These take data sets for k groups and level of significance as input variables. To apply this package for $k = 3$ or $k = 4$, one needs to run the library 'library(CIEXP3)' or 'library(CIEXP4)' followed by installing the package with the command 'install_github("TulikaStat/CIEXP3")' or 'install_github("TulikaStat/CIEXP4")' respectively. This package takes input data sets

Table 10. CP and AL of CI for P_4: $k = 4$, $n_1 = 30, n_2 = 25, n_3 = 20, n_4 = 15$ and $\eta_1 = 1, \eta_3 = 2$

η_2	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.942	0.943	0.943	0.876	0.885	0.1483	0.1457	0.1427	0.1378	0.1198
0.4	0.937	0.945	0.944	0.875	0.877	0.1615	0.1534	0.157	0.1509	0.1285
0.6	0.941	0.946	0.957	0.898	0.858	0.1711	0.1692	0.1658	0.1575	0.1344
0.8	0.941	0.942	0.944	0.868	0.885	0.1772	0.1774	0.1725	0.1667	0.1412
1	0.91	0.938	0.943	0.895	0.862	0.1803	0.1842	0.1769	0.1727	0.1463
1.2	0.938	0.941	0.951	0.882	0.886	0.1859	0.1853	0.1791	0.1759	0.1503
1.4	0.947	0.943	0.948	0.882	0.889	0.187	0.1874	0.1809	0.1775	0.1531
1.6	0.934	0.928	0.928	0.881	0.89	0.1875	0.1893	0.1817	0.1789	0.1577
1.8	0.95	0.942	0.939	0.904	0.912	0.1875	0.1904	0.1831	0.1811	0.1618
2	0.938	0.931	0.942	0.891	0.91	0.1882	0.1903	0.1841	0.182	0.1649
2.2	0.937	0.957	0.943	0.886	0.919	0.1886	0.1912	0.1839	0.1822	0.1683
2.4	0.95	0.946	0.939	0.889	0.918	0.1874	0.1904	0.1833	0.1824	0.1719
2.6	0.95	0.946	0.941	0.875	0.933	0.1874	0.191	0.1827	0.1809	0.1748
2.8	0.947	0.948	0.939	0.88	0.912	0.1871	0.1896	0.1818	0.1815	0.1768
3	0.96	0.951	0.952	0.892	0.923	0.1856	0.1887	0.1811	0.1812	0.1817
3.2	0.95	0.934	0.939	0.893	0.937	0.1851	0.1862	0.1807	0.181	0.1837
3.4	0.941	0.942	0.943	0.887	0.935	0.184	0.1863	0.1796	0.1808	0.1849
3.6	0.94	0.931	0.948	0.874	0.933	0.1838	0.1847	0.1796	0.1795	0.1877
3.8	0.958	0.952	0.95	0.849	0.956	0.1829	0.1844	0.1792	0.1782	0.1915
4	0.939	0.936	0.937	0.9	0.953	0.1816	0.1842	0.1775	0.1784	0.1952

Table 11. CP and AL of CI for P_1: $k = 4$, $n_1 = 5, n_2 = 5, n_3 = 6, n_4 = 10$ and $\eta_1 = 0.5, \eta_2 = 0.5$

η_3	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.944	0.944	0.935	0.644	0.897	0.2819	0.2959	0.2708	0.2242	0.2453
0.4	0.947	0.928	0.931	0.676	0.894	0.3026	0.3155	0.2857	0.2481	0.2693
0.6	0.944	0.946	0.94	0.676	0.9	0.3097	0.3285	0.2943	0.2616	0.2797
0.8	0.931	0.949	0.935	0.636	0.901	0.3112	0.3264	0.295	0.2659	0.2897
1	0.932	0.948	0.939	0.646	0.931	0.3114	0.3319	0.2951	0.2691	0.3004
1.2	0.949	0.947	0.937	0.645	0.924	0.3091	0.3293	0.2955	0.734	0.3047
1.4	0.935	0.934	0.928	0.667	0.927	0.3081	0.3267	0.2918	0.2762	0.3119
1.6	0.936	0.941	0.916	0.644	0.942	0.3067	0.3112	0.2906	0.2756	0.2069
1.8	0.93	0.946	0.937	0.634	0.938	0.3057	0.3215	0.2909	0.275	0.3245
2	0.945	0.941	0.93	0.639	0.942	0.3002	0.3191	0.2868	0.2744	0.3278
2.2	0.936	0.941	0.941	0.634	0.958	0.2994	0.3163	0.2855	0.2733	0.3378
2.4	0.947	0.94	0.935	0.655	0.954	0.2956	0.3141	0.2831	0.2735	0.339
2.6	0.933	0.937	0.947	0.63	0.968	0.2937	0.3102	0.2827	0.2712	0.3437
2.8	0.945	0.937	0.938	0.614	0.959	0.2915	0.3077	0.2791	0.2686	0.3466
3	0.955	0.94	0.957	0.644	0.959	0.2911	0.3059	0.279	0.2689	0.3522
3.2	0.951	0.941	0.931	0.638	0.967	0.2894	0.3032	0.2808	0.2673	0.3521
3.4	0.937	0.949	0.94	0.63	0.97	0.2856	0.301	0.2758	0.2664	0.3519
3.6	0.95	0.936	0.95	0.637	0.974	0.2841	0.2967	0.2674	0.2657	3562
3.8	0.953	0.943	0.94	0.615	0.975	0.2842	0.2961	0.2678	0.2637	0.3672
4	0.946	0.943	0.92	0.61	0.978	0.2794	0.2929	0.273	0.2618	0.3663

for $X_{11}, X_{12}, \ldots, X_{1n_1}, \ldots, X_{k1}, \ldots, X_{kn_k}$ and level of significance α. The data sets can be in '.csv' format or they can be directly imported to 'R' script. Under this package, output comes as 'estimated.value', 'lower.bound', 'upper.bound' and 'length' for MLE, UMVUE, GB and BSE respectively for $P_1, P_2, \ldots, P_k$.

4. Real Data Example

There are large number of modern diseases which are attributed to a stressful and unhealthy lifestyle. To overcome these, medical practitioners and dietitians recommend food items with low fat. We use a data set on 'fat content' on various edible items collected during Covid-2019 and now made available

Table 12. CP and AL of CI for P_2: $k = 4$, $n_1 = 5, n_2 = 5, n_3 = 6, n_4 = 10$ and $\eta_1 = 0.5, \eta_2 = 0.5$

η_3	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.945	0.956	0.946	0.655	0.958	0.3077	0.3276	0.2925	0.2937	0.4049
0.4	0.937	0.937	0.932	0.661	0.944	0.3167	0.3362	0.299	0.3147	0.4176
0.6	0.945	0.943	0.94	0.622	0.969	0.3149	0.3353	0.3006	0.3186	0.4333
0.8	0.953	0.938	0.947	0.645	0.96	0.312	0.3272	0.2961	0.3211	0.4422
1	0.947	0.932	0.936	0.663	0.954	0.3058	0.3248	0.2899	0.3196	0.4477
1.2	0.935	0.935	0.955	0.646	0.966	0.2986	0.3194	0.2886	0.3177	0.457
1.4	0.953	0.95	0.931	0.62	0.974	0.2965	0.3149	0.2823	0.3151	0.465
1.6	0.951	0.94	0.953	0.6	0.967	0.2938	0.312	0.2787	0.3114	0.4636
1.8	0.954	0.947	0.954	0.629	0.975	0.2896	0.3046	0.2778	0.3079	0.4677
2	0.934	0.953	0.942	0.602	0.971	0.2831	0.3015	0.2736	0.3055	0.4789
2.2	0.946	0.952	0.948	0.565	0.973	0.2821	0.298	0.2716	0.3039	0.4768
2.4	0.948	0.942	0.946	0.596	0.975	0.2781	0.295	0.2676	0.3025	0.4786
2.6	0.949	0.949	0.953	0.596	0.97	0.2769	0.2916	0.2679	0.2993	0.4784
2.8	0.946	0.946	0.947	0.577	0.987	0.2915	0.2899	0.2649	0.2979	0.4853
3	0.957	0.946	0.952	0.579	0.992	0.2738	0.2866	0.2629	0.2945	0.4857
3.2	0.96	0.93	0.96	0.604	0.982	0.2715	0.2836	0.2621	0.2936	0.4908
3.4	0.95	0.945	0.951	0.592	0.985	0.2681	0.2833	0.2597	0.2888	0.4953
3.6	0.945	0.941	0.917	0.564	0.989	0.2662	0.2815	0.2528	0.2897	0.4965
3.8	0.954	0.937	0.927	0.577	0.985	0.2666	0.2826	0.2589	0.2873	0.4977
4	0.951	0.952	0.952	0.575	0.989	0.2631	0.2805	0.2592	0.2852	0.4993

Table 13. CP and AL of CI for P_3: $k = 4$, $n_1 = 5, n_2 = 5, n_3 = 6, n_4 = 10$ and $\eta_1 = 0.5, \eta_2 = 0.5$

η_3	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.942	0.954	0.955	0.643	0.953	0.2902	0.3094	0.2775	0.2787	0.3896
0.4	0.952	0.942	0.949	0.633	0.938	0.3009	0.3172	0.2856	0.2956	0.4094
0.6	0.953	0.947	0.957	0.651	0.97	0.3001	0.3177	0.2847	0.3012	0.4195
0.8	0.95	0.945	0.951	0.593	0.966	0.2959	0.3108	0.2814	0.2999	0.4303
1	0.952	0.943	0.947	0.63	0.954	0.2897	0.3081	0.2763	0.299	0.4344
1.2	0.944	0.941	0.943	0.623	0.971	0.2859	0.3029	0.2728	0.2972	0.4445
1.4	0.948	0.944	0.934	0.604	0.961	0.2824	0.2975	0.2686	0.2962	0.4471
1.6	0.946	0.942	0.946	0.63	0.964	0.2786	0.2945	0.2669	0.2928	0.4573
1.8	0.946	0.941	0.944	0.608	0.97	0.2748	0.2902	0.2647	0.2887	0.4639
2	0.942	0.951	0.944	0.615	0.97	0.2707	0.2879	0.2596	0.287	0.4648
2.2	0.946	0.944	0.934	0.644	0.977	0.2693	0.2848	0.2582	0.2854	0.4681
2.4	0.946	0.957	0.944	0.618	0.988	0.2667	0.2817	0.2553	0.2825	0.4766
2.6	0.933	0.954	0.963	0.623	0.991	0.2633	0.2787	0.2536	0.2802	0.4813
2.8	0.95	0.952	0.951	0.608	0.983	0.2626	0.2768	0.251	0.2778	0.475
3	0.948	0.95	0.941	0.608	0.987	0.2589	0.2745	0.2529	0.2762	0.4811
3.2	0.962	0.946	0.942	0.589	0.987	0.2594	0.2713	0.2586	0.2735	0.4867
3.4	0.942	0.945	0.932	0.587	0.992	0.2568	0.2684	0.2486	0.2715	0.4887
3.6	0.986	0.941	0.932	0.605	0.986	0.2555	0.2698	0.2465	0.2708	0.4941
3.8	0.988	0.934	0.919	0.607	0.988	0.2539	0.2699	0.2427	0.2683	0.4849
4	0.979	0.938	0.964	0.589	0.979	0.2516	0.2681	0.2491	0.268	0.4903

on "https://www.kaggle.com/datasets/mariaren/covid19-healthy-diet-dataset". Here from the entire set of items, we have selected items in four food groups, viz. 'pulses', 'tree nuts', 'animal fats' and 'stimulants'. First we carry out goodness of fit tests to confirm that these four data sets are from exponential distributions with different scale parameters. The values of test statistic using chi-square test with eight intervals are seen to be $10.3, 7.1, 14.1$ and 8.8. Next, we take independent random samples of sizes $n_1 = 5$, $n_2 = 5$, $n_3 = 7$ and $n_4 = 9$ respectively from the four groups on 'pulses', 'tree nuts', 'animal fats' and 'stimulants'. We have taken small samples so that the relative differences in values of estimates and confidence limits based on different procedures are clearly reflected. The data set is given in Table 15.

The values of point estimates and 95% confidence bounds for P_1, P_2, P_3 and P_4 using $\delta_{i,M}$, $\delta_{i,U}$, $\delta_{i,GB}$ and $\delta_{i,BS}$, $i = 1, 2, 3, 4$ are given in Tables 16-20. We note that values

Table 14. CP and AL of CI for P_4: $k = 4$, $n_1 = 5, n_2 = 5, n_3 = 6, n_4 = 10$ and $\eta_1 = 0.5, \eta_2 = 0.5$

η_3	Coverage probability					Average length				
	MLE	UMVUE	GB	BSE	AMLE	MLE	UMVUE	GB	BSE	AMLE
0.2	0.92	0.953	0.933	0.61	0.997	0.2223	0.2279	0.2139	0.247	0.5872
0.4	0.928	0.945	0.94	0.637	0.964	0.2586	0.2699	0.2459	0.2929	0.4369
0.6	0.945	0.944	0.939	0.689	0.946	0.2634	0.2787	0.2506	0.2562	0.5515
0.8	0.947	0.928	0.945	0.727	0.904	0.2581	0.2719	0.2477	0.2418	0.2961
1	0.938	0.956	0.944	0.74	0.902	0.249	0.2656	0.2402	0.2261	0.2583
1.2	0.959	0.939	0.954	0.723	0.899	0.2406	0.2536	0.2327	0.2137	0.2277
1.4	0.959	0.953	0.947	0.733	0.873	0.2317	0.2439	0.2242	0.2027	0.2069
1.6	0.936	0.948	0.951	0.707	0.852	0.2225	0.2367	0.2158	0.1907	0.1861
1.8	0.935	0.937	0.941	0.708	0.857	0.2143	0.2219	0.2115	0.1793	0.1715
2	0.952	0.933	0.955	0.73	0.846	0.2036	0.2139	0.2024	0.1716	0.1614
2.2	0.951	0.935	0.951	0.717	0.84	0.1977	0.2052	0.1967	0.164	0.1484
2.4	0.961	0.941	0.952	0.718	0.828	0.1898	0.1945	0.1899	0.1569	0.1401
2.6	0.936	0.942	0.946	0.7	0.825	0.1822	0.1882	0.1849	0.1483	0.132
2.8	0.944	0.944	0.936	0.696	0.815	0.1791	0.1833	0.1789	0.1433	0.1277
3	0.952	0.937	0.95	0.696	0.813	0.1712	0.1767	0.175	0.137	0.1207
3.2	0.942	0.944	0.96	0.692	0.81	0.1671	0.1712	0.1778	0.1327	0.1147
3.4	0.929	0.944	0.95	0.681	0.82	0.1597	0.1622	0.1638	0.1263	0.1116
3.6	0.945	0.935	0.96	0.701	0.818	0.154	0.1508	0.1587	0.1226	0.1056
3.8	0.941	0.943	0.942	0.693	0.801	0.1517	0.1492	0.1491	0.1189	0.1003
4	0.955	0.93	0.941	0.684	0.817	0.1465	0.1449	0.1534	0.1151	0.0971

Table 15. Fat content data set

Pulses:	0.6535	0.0518	0.1744	2.6909	0.0689
Tree nuts:	0.8805	0.7817	0.128	0.4175	0.0315
Animal fats:	0.8839	1.7622	3.4811	2.6611	1.7819
	4.0036	10.0008			
Stimulants:	0.0984	0.3833	3.3838	0.9894	0.6279
	0.0318	0.8305	0.0671	0.7849	

of $\delta_{i,M}$, $\delta_{i,U}$, $\delta_{i,GB}$ are quite close, but those of $\delta_{i,BS}$ are different. The values of estimates establish that the fat content in 'tree nuts' is the least and less than that of the average distribution, whereas for 'animal fats' it is the highest and more than that of the average distribution. The confidence bounds based on MLE, UMVUE and GB support this. Similarly, for 'pulses' and 'stimulants', the fat content is close to that of the average distribution and the confidence limits support this as the limits include 0.5.

Table 16. Estimates of SSI for the Fat Content Data

P_i	$\delta_{i,M}$	$\delta_{i,U}$	$\delta_{i,GB}$	$\delta_{i,BS}$
P_1	0.5581	0.5554	0.5525	0.4144
P_2	0.6617	0.6690	0.6499	0.5192
P_3	0.2425	0.2321	0.2545	0.1731
P_4	0.5374	0.5433	0.5429	0.4495

Table 17. Confidence interval for SSI P_1

Estimators	Lower bound	Upper bound	Length
MLE	0.4084	0.7116	0.3031
UMVUE	0.3992	0.7228	0.3236
GB	0.4095	0.6996	0.2901
BSE	0.2816	0.5838	0.3022

Table 18. Confidence interval for SSI P_2

Estimators	Lower bound	Upper bound	Length
MLE	0.5061	0.7848	0.2786
UMVUE	0.5012	0.7942	0.2931
GB	0.5021	0.7723	0.2702
BSE	0.3683	0.6798	0.3115

Table 19. Confidence interval for SSI P_3

Estimators	Lower bound	Upper bound	Length
MLE	0.1751	0.3524	0.1773
UMVUE	0.1691	0.3435	0.1744
GB	0.1828	0.3634	0.1806
BSE	0.1279	0.2544	0.1264

Table 20. Confidence interval for SSI P_4

Estimators	Lower bound	Upper bound	Length
MLE	0.4091	0.6521	0.2430
UMVUE	0.4096	0.6641	0.2544
GB	0.4197	0.6521	0.2323
BSE	0.3336	0.5693	0.2356

5. Conclusion

In this paper, a new stress-strength index introduced in Gupta et al. [14] is considered for one parameter exponential distributions from k groups. Parametric bootstrap confidence intervals are constructed using percentile method based on the MLE, UMVUE, GB and BSE and their estimated coverage probabilities and average lengths are compared with the confidence intervals based on the asyptotic distribution of the MLE. Numerical comparisons suggest that p-bootstrap confidence intervals based on a generalized Bayes estimator have the best performance for small, moderate and large sample sizes. For moderate and large sample sizes, the procedures based on the MLE and UMVUE also perform quite well. Two 'R' packages are developed for implementation of the procedures when number of groups is 3 or 4. These packages are shared on open source platform 'GitHub'. Finally, all the procedures are illustrated on a real data set.

Acknowledgement(s)

We are thankful to the Editor-in-Chief and the reviewers for their valuable suggestions which have led to substantial improvement in the content and presentation of the paper.

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